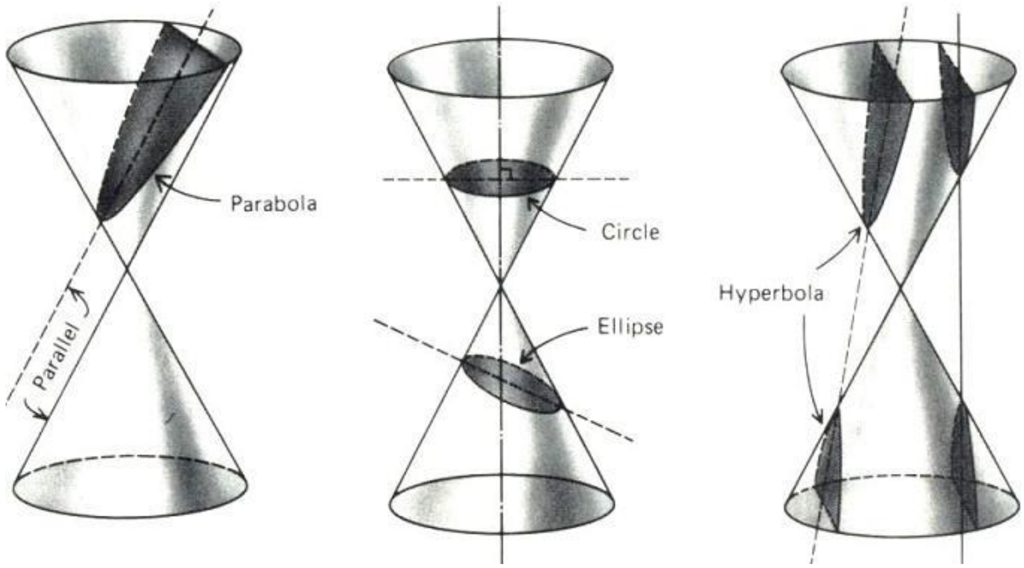
Conic Sections

Circles, parabolas, ellipses, and hyperbolas are intersections of a plane with a double cone as shown in the diagram below.



Standard equations in rectangular coordinates are found using definitions involving a center, focus and directrix (parabola), or two foci (ellipse and hyperbola). The equations below are correct provided the center (or in the case of the parabola, the vertex) is at the Origin point (0, 0).

I. Circles: $x^2 + y^2 = r^2$ where r is the radius of the circle.

II. (1) Parabolas: $x^2 = 4cy$ $|c|$ is the distance from the vertex to the focus (and from the vertex to the "directrix" which is a line). This parabola opens up or down depending upon the sign of c.

$y^2 = 4cx$ As above, but this parabola opens left or right based on the sign of c.

(2) Reflective properties: any signal originating from the focus of the parabola reflects off the interior of the parabola and continues parallel to the axis of the parabola; also signals coming into a parabola reflect through the focus. Applications – headlights, satellite dishes, telescopes, solar cooking

III. (1) Ellipses: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

a and b are the distances from the center left/right and up/down to the extremities of the ellipse. These also give the x-intercepts and y-intercepts.

The foci are found on the major axis, determined by which is larger, a or b. We let c be the focal distance (from the center to the focus). Then if $b > a$, $c=\sqrt{b^2-a^2}$ and if $a > b$, then $c=\sqrt{a^2-b^2}$.

(2) Reflective properties: a signal emanating from one focus reflects off the interior of the ellipse and passes through the other focus. Application – gallstone surgery: see more about this at http://mathcentral.uregina.ca/beyond/articles/Lithotripsy/lithotripsy1.html

IV. (1) Hyperbolas: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ opening left/right or $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ opening up/down

The branches of a hyperbola open within asymptotes containing the Origin; slopes are $\pm\frac{b}{a}$.

The distance from the center to the foci is given by $c=\sqrt{a^2+b^2}$.

(2) Reflective properties: a signal aiming toward a focus reflects off the exterior (convex side) of one branch toward the other focus. Application: Cassegrain reflecting telescope wherein a parabolic mirror and a convex branch of a hyperbola share a focus; light entering the telescope reflects off the parabolic mirror toward the shared focus, but before reaching this focus reflects off the convex hyperbola toward the other hyperbolic focus located at the eyepiece.

V. Examples of ellipse, parabola and hyperbola graphs, equations, and locations of foci:

(1) Ellipse example at right:

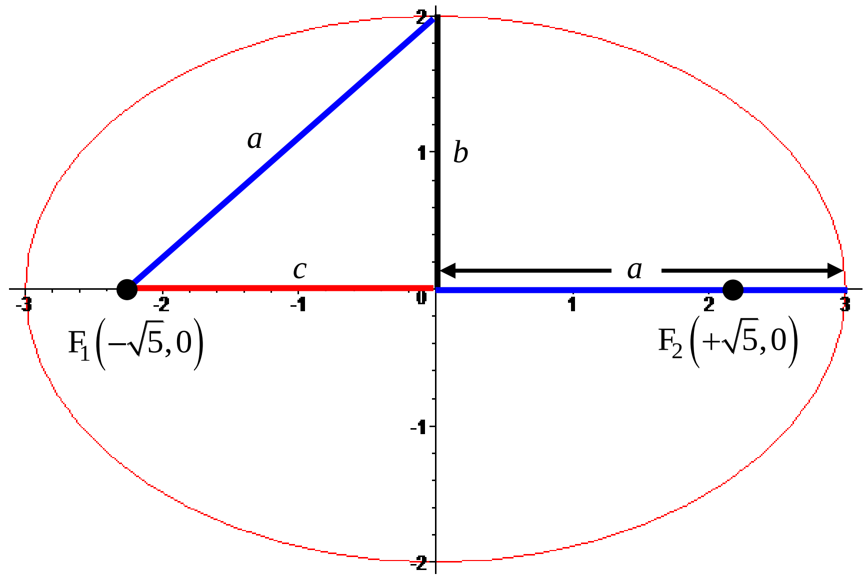
$4x^2 + 9y^2 = 36$

$\rightarrow \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

$a = 3$ and $b = 2$

$\rightarrow c = \sqrt{a^2 - b^2} = \sqrt{5}$

F_1 and F_2 are the foci



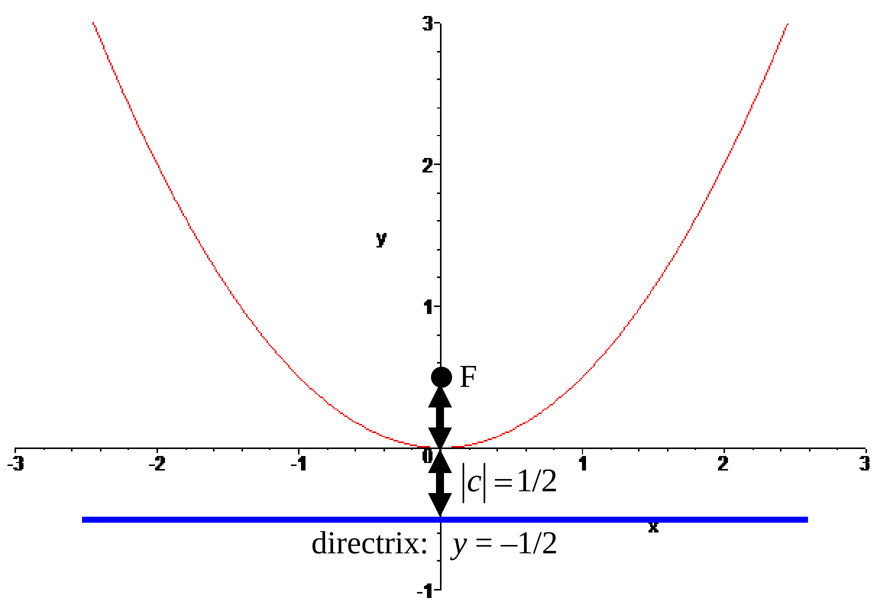
(2) Parabola example at right:

$y = \dfrac{1}{2}x^2 \rightarrow 2y = x^2$

Use the form $x^2 = 4cy$:

Thus $4c = 2 \rightarrow c = \dfrac{1}{2}$

The focus F is the point (0, 1/2).



(3) Hyperbola example at right below:

$$4x^2 - 9y^2 = 36$$
$$\to \frac{x^2}{9} - \frac{y^2}{4} = 1$$
$$a = 3 \text{ and } b = 2$$
$$\text{slope of asymptotes} = \pm\frac{b}{a} = \pm\frac{2}{3}$$
$$\to c = \sqrt{a^2 + b^2} = \sqrt{13}$$
$$F_1 : \left(-\sqrt{13}, 0\right) \text{ and } F_2 : \left(\sqrt{13}, 0\right) \text{ are the foci}$$